PERGAMON

International Journal of Solids and Structures 37 (2000) 3345–3373

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Some contact problems of anisotropic elastodynamics: integral characteristics and exact solutions

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Received 27 October 1998; in revised form 6 April 1999

Abstract

The author's method of integral characteristics of solutions to boundary-initial value problems of contact is applied to solve some new problems of anisotropic elastodynamics. For both anisotropic elastic and elastic with initial stresses media, solutions are obtained to impact problems which were solved earlier for only an isotropic linear elastic medium. It is also found the resultant contact force in the problems of both frictional and frictionless pressing normal to the boundary plane of the anisotropic continuous both an elastic with initial stresses and a linear viscoelastic orthotropic half-spaces. Some general properties of dynamical contact problems are also formulated. In particular, conditions are provided when problems of vertical pressing are self-similar. Crown Copyright © 2000 Published by Elsevier Science Ltd. All rights reserved.

Keywords: Impact; Collision; Wave propagation; Friction

1. Introduction

Dynamic contact problems for continuous media are mathematically very complicated and usually the use of traditional methods does not allow us to find the transient fields of displacements $\mathbf{u}(\mathbf{x}, t)$ and stresses $\sigma_{ij}(\mathbf{x}, t)$ arising in the media during the contact. However, sometimes it is possible to find some integral characteristics of these fields without their study in detail. Examples of such characteristics are the resultant force P of contact between the medium and the contacting body and the resulting moments M_i of contacting stresses.

To study these quantities the author introduced the so-called method of integral characteristics of solutions to boundary-initial value problems of contact (Borodich, 1990a, 1990b, 1990c, 1991). The key

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PII: S0020-7683(99)00135-3

idea of the method is to reduce the problem for the integral characteristics of a three-dimensional problem to a problem of plane waves propagation in the same medium. The results of the above author's papers were summed up as:

> It has been developed a method to study integral characteristics of solutions to dynamical problems of pressing of blunt dies into both homogneous and non-homogneous along depth media with linear constitutive relations. The method can be applied to both problems of pressing normal to boundary plane of the continuous half-space and to problems of pressing with rotation of body axes. Using this method, there has been obtained both expressions for the resultant forces, which affect the body during the dynamical pressing, and expressions for the resulting moments of reacting stresses (in the case of orthotropic media). The method's means has been shown by giving examples of anisotropic linear elastic, orthotropic viscoelastic, non-homogneous along depth isotropic linear elastic, and elastic with initial stresses media

(see p. 256 in Borodich 1990c).

The method is quite general. Indeed, the integral characteristics for anisotropic elastic media were found in the above author's papers for the first time. However, it is necessary to underline that the method (Borodich, 1990a, 1990b, 1990c, 1991) was applied only to frictionless contact problems. Here, we continue to show the method's means by solving new dynamical contact problems and problems of collision, in particular frictional problems for both anisotropic elastic and elastic with initial stresses media.

The paper is organised as follows: in Section 2, some preliminary information attributed to dynamical contact problems is recalled. In Section 3, both a piecewise smooth formulation and a generalised weak formulation of the dynamical problems are provided. In Section 4, it is shown that the problems of vertical pressing usually have the supersonic stage of the contact region extension. Here, conditions are also provided when problems of vertical pressing are self-similar. The exact solutions to plane wave problems for anisotropic elastic and elastic with initial stresses media are given in in Section 5. Here, the general form of solutions to plane wave problems for orthotropic viscoelastic solids is also given. The method of integral characterisics is described in Section 6. Integral characteristics for frictionless problems of pressing a die with rotation of its axes into a prestressed elastic half-space are found in Section 7. Finally, in Section 8, the first stage of frictional collision of a die and a continuous anisotropic half-space is studied. The die is treated there as a blunt convex rigid body having an arbitrary shape with two orthogonal planes of symmetry, which are both orthogonal to the boundary of half-space. Exact expressions are obtained for the relationships between time, depth of indentation, and velocity of the body. A proof is given that the expressions are independent of the boundary conditions in the contact region.

2. Preliminaries

Here, we will give some information concerning dynamical contact problems.

2.1. Self-similar problems of elastodynamics

Due to their complexity, the dynamic problems of indentation of a rigid die into a continuous half-space were studied initially in cases of self-similar problems for linear elastic media. To ensure the self-similar character of the contact problems the die velocity must be given in advance (see, for example, Borodich, 1988a, 1990c). There are several approaches for self-similar problems of elastodynamics, namely the Smirnov–Sobolev method (Smirnov and Sobolev, 1932, 1933; Sobolev, 1937) of functional-invariant solutions, the Willis (1973) approach based on the transform methods and two other methods, both being introduced by Brock: the homogeneous-function method (Brock, 1976, 1977, 1978) and integral-transform method (Brock, 1993).

The Smirnov–Sobolev method was originally intoduced for two-dimensional problems. However, Kostrov (1964a, 1964b) developed it and showed that the Smirnov–Sobolev method can be used to solve some axisymmetrical self-similar problems. Using his method of superposition of functional-invariant solutions, Kostrov (1964a, 1964b) solved the self-similar problems of frictionless pressing of rigid conical and wedge-shaped dies into an elastic half-space. Note that Kostrov proposed a few additional hypotheses concerning regions of contact. The same problems, without these hypotheses, were solved independently by Afanasev and Cherepanov (1973) and Robinson and Thompson (1974a, 1974b).

It is known that the Smirnov–Sobolev method is effective when there are no tangental stresses on the boundary of the half-space and the medium is isotropic. Willis (1973) developed another method and studied self-similar problems of frictionless and adhesive pressing of wedge-shaped dies into an elastic anisotropic half-space. Then the Willis method was successfully used to solve boundary-initial value problems with both frictionless and frictional boundary conditions (see, e.g., Bedding and Willis, 1973, 1976). The Brock approaches were developed in application to various frictional self-similar contact poblems by Brock and Georgiadis, 1994 and Georgiadis et al., 1995.

The self-similar problems include the problems when the traction vector $\mathbf{T}$ or displacements $\mathbf{u}^*$ known on the boundary plane are homogeneous functions, in particular

$$\mathbf{T} = \delta(t)\delta(x_1)\delta(x_2)\mathbf{i}_l, \ \mathbf{u}^* = \delta(t)\delta(x_1)\delta(x_2)\mathbf{i}_l \tag{1}$$

where δ is Dirac delta and $\mathbf{i}_l$ is the unit vector directed along the x_l-axis. The solutions to these problems can be used as the Green functions for transient elastodynamic problems.

2.2. Integral characteristics for problems of vertical indentation

Let a homogeneous half-space $\mathbb{R}^3_+$, initially at rest, be subject to dynamic indentation by a blunt convex die, whose shape is defined by the graph of the non-negative function $f(x_1, x_2)$.

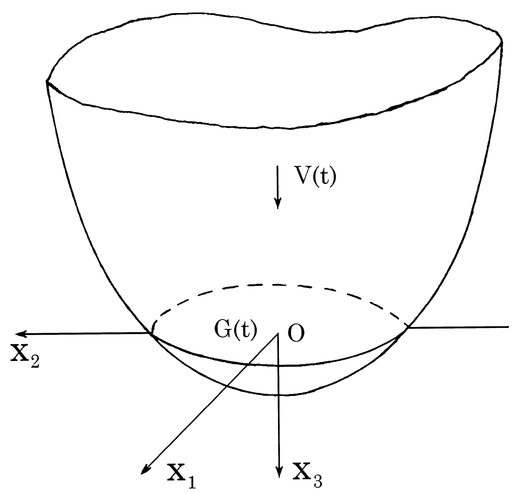

Fig. 1. A sketch of the die and the coordinate axes (after Borodich, 1990c).

Let the velocity of the die ($V(t)$) be directed along the normal to the boundary of the half-space $x_3 = 0$ (Fig. 1). Let $G(t)$ be the open region of contact between the die and the medium, S the area of G, ∂G its boundary. Let γ be the speed of the curve ∂G along the boundary of the half-space $\mathbb{R}^3_+$, measured along the normal to ∂G. We will show below that if the initial velocity $V(0)$ of the die is non zero, then there exists a time interval $[0, t_a]$, $t_a > 0$, on which γ exceeds the maximum speed of wave propagation in the medium. It is said that the process of indentation in the interval $[0, t_a]$ is supersonic in nature.

In the dynamical problems with *a priori* given velocity of a rigid die, an extremely simple relationship for the instantaneous value of the force $P(t)$, required to indent the die during the supersonic stage of contact was obtained: the force $P(t)$ is directly proportional to the product of the velocity of indentation $V(t)$ and the area $S(t)$ of contact

$$P(t) = \rho a V(t) S(t). \tag{2}$$

Here, ρ is the density of the medium and a is the constant speed of sound in fluid or the speed of propagation of the longitudinal waves in elastic body.

First, this relationship Eq. (2) was obtained in the problem of contact for the ideal compressible fluid (acoustic medium) by Skalak and Feit (1966). Then the validity of the relationship (Eq. (2)) in the problem of frictionless contact between an axisymmetrical die and an isotropic elastic half-space was announced by Simonov and Flitman (1966) and the full proof was published by Simonov (1967). The validity of the relationship in the frictionless three-dimensional problem for the isotropic elastic medium was shown by Robinson and Thompson (1975). The relationship for the contact force $P(t)$, which was obtained by Bedding and Willis (1976) from the exact solution to the adhesive self-similar contact problem, conforms precisely to Eq. (2).

After the end of the supersonic stage, the value of the vector of displacements $\mathbf{u}(x_1, x_2, 0, t)$ is different from zero only over the bounded region U, which has the decomposition $U = G \cup G_1$, where G_1 is the region of disturbed motion of the boundary surface particles, which are not in mutual contact with the die. It was found by Skalak and Feit (1966) that

$$P(t) = \rho a [V(t) S(t) + V_1(t) S_1(t)], \tag{3}$$

where $V_1(t)$ and $S_1(t)$ are the mean velocity over the region G_1 and the area of this region, respectively. Then it was shown that the relationship Eq. (3) is valid in the problems of frictionless indentation for isotropic (Popov, 1990) and anisotropic elastic media (Borodich, 1990a, 1990b) and in the case of frictional indentation (Borodich, 1995).

It should be noted that various techniques were used to obtain the relationships Eq. (2). To study the problems, Skalak and Feit (1966) employed the method of the retarded potential, Simonov used the Hankel transform, while Kubenko and Popov (1989) and Popov (1990) (see also Kubenko, 1997) used the Laplace transform and the Fourier–Bessel series expansion. Robinson and Thompson (1975) used an indirect way. Using the technique of the Smirnov–Sobolev functional-invariant solutions, they found a solution to an auxiliary self-similar problem when velocities at the boundary points are described as the concentrated velocity, i.e., formulae for velocities are similar to the formulae in Eq. (1). Then they solved the main problem by integration of the obtained auxiliary solutions.

There arises a natural question:

> Why was the superposition of the Willis solutions for self-similar problems of concentrated velocities or loads not used in order to find the integral characteristics?

(Professor B.E. Pobedrya, Moscow State University, 1989)

The answer is that the Willis method is very general and it allows us to find the detailed distribution of both stress and displacement fields in elastic media. However, the integral characteristics can be found in a simpler way. On the other hand, the applicability of the Willis' method is restricted by elastic media only. As mentioned above, the author's method was applied to various media whose properties cannot be completely described by Hooke's law (Borodich, 1990c, 1991).

3. Formulations of elastodynamic problems

We choose the origin of a Cartesian coordinate system $Ox_1x_2x_3$ to be at the point of initial contact between the die and the half-space. We orient the x_3 axis into the interior of the half-space and the x_1 and x_2 axes along its boundary (Fig. 1).

3.1. Classical equations and piecewise smooth solutions

We will give the formulation of the problem for the case of linear anisotropic elastodynamics keeping in mind that the formulation of the problem for homogeneously prestressed elastic media can be obtained from this case by substituting y_k instead of x_k, ω^*_{ijkl} instead of C_{ijkl} and Q_{ij} instead of σ_{ij} (see Appendix A).

Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement vector of the particles of the medium, $\sigma_{ij}(\mathbf{x}, t)$ the components of the stress tensor associated with the disturbance produced by the die. The force of interaction between the body and the medium is defined by

$$P(t) = \int\int_{G(t)} \sigma_{33}(x_1, x_2, 0, t)\, \mathrm{d}x_1\, \mathrm{d}x_2.$$

If we seek the solution to a linearized problem of elastodynamics between piecewice regular elastodynamic states on $\mathbb{R}^3_+ \times [0, t_*]$, then there exists a regular partition $\{D_1, D_2,\ldots, D_m\}$ of $\mathbb{R}^3_+ \times [0, t_*]$ separated by surfaces $\mathscr{S}_k$, which are surfaces of discontinuity for derivatives of the displacement vector.

Let us write equations in the domains where the vector $\mathbf{u}$ is continuous together with its derivatives up to second order with respect to the coordinates and time. Then we have the following classical elastodynamic equations of motion

$$\sigma_{ji,\,j}(\mathbf{x}, t) - \rho\ddot{u}_i(\mathbf{x}, t) = 0, \quad i, j = 1, 2, 3; \tag{4}$$

and constitutive relations

$$\sigma_{ij} = \mathscr{F}\left\{\frac{(u_{k,\,l} + u_{l,\,k})}{2}\right\} \quad \text{on} \quad \mathbb{R}^3_+ \times [0, t]. \tag{5}$$

Here and henceforth, the dot will denote the derivative with respect to time and a comma before the subscript will denote the derivative with respect to the corresponding coordinate; summation from 1 to 3 is assumed over repeated Latin subscripts, while there is no summation over the Greek subscripts. $\mathscr{F}$ is the operator of the constitutive relations. For anisotropic elastic media, the constitutive relations have the following form

$$\sigma_{ij} = C_{ijkl}u_{k,\,l}, \quad C_{ijkl} = C_{jikl} = C_{klij}. \tag{6}$$

The tensor C_{ijkl} is positive-definite (see, e.g., Guz, 1986a). In particular, in the case of an isotropic elastic medium, Hooke's law becomes

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,\,k} + \mu(u_{i,\,j} + u_{j,\,i}),$$

where λ and μ are the Lamé coefficients.

These $\mathbf{u}$ and σ_{ij} should satisfy the following initial conditions of a body at rest and boundary conditions of contact

$$\mathbf{u}(\mathbf{x}, 0) = \dot{\mathbf{u}}(\mathbf{x}, 0) = 0,$$

$$u_3(x_1, x_2, 0, t) = g(x_1, x_2, t), \quad (x_1, x_2) \in G(t)$$

$$\sigma_{3i}(x_1, x_2, 0, t) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash G(t), \tag{7}$$

where $g(x_1, x_2, t)$ is a known function. In the problem of vertical pressing, we have

$$g(x_1, x_2, t) = H(t) - f(x_1, x_2), \quad (x_1, x_2) \in G(t), \tag{8}$$

where $H(t)$ is the depth of indentation of the die apex, which is given by the formula

$$H(t) = \int_0^t V(\tau)\mathrm{d}\tau. \tag{9}$$

To complete the formulation of the boundary-initial value problem it is necessary to give two additional conditions for tangential components of stresses or displacements within $G(t)$.

Let us denote

$$\mathbf{W}(x_1, x_2, t) \equiv \{u_1(x_1, x_2, 0, t), u_2(x_1, x_2, 0, t)\},$$

$$\boldsymbol{\tau}(x_1, x_2, t) \equiv \{\sigma_{31}(x_1, x_2, 0, t), \sigma_{32}(x_1, x_2, 0, t)\}.$$

The frictionless contact problem should satisfy the following conditions

$$\boldsymbol{\tau}(x_1, x_2, t) = 0, \quad (x_1, x_2) \in G(t). \tag{10}$$

In the case of full adhesion, there is no relative slip between the die and the boundary of the half-space within the contact region, i.e., the tangential components of displacements within $G(t)$ cannot change with augmentation of the indentation depth. This is expressed by

$$\frac{\partial W_1(x_1, x_2, t)}{\partial t} = \frac{\partial W_2(x_1, x_2, t)}{\partial t} = 0, \quad (x_1, x_2) \in G(t). \tag{11}$$

It is usually assumed in the frictional contact problems that the contact region G consists of the following parts:

(i) the inner adhesive part, G_A, where the interfacial friction is sufficient to prevent any slip between the die and the half-space, i.e., (Eq. (11)) holds;
(ii) the outer adhesive part, G_F, where the interfacial friction satisfies the Coulomb frictional law:

$$\boldsymbol{\tau}(x_1, x_2, t) = -\phi\sigma_{33}(x_1, x_2, 0, t)\left[\frac{\mathbf{W}(x_1, x_2, t)}{|\mathbf{W}(x_1, x_2, t)|}\right], \quad (x_1, x_2) \in G_F(t), \tag{12}$$

where ϕ is the coefficient of friction.

The above equations give the formulation of a boundary-initial value contact problem in the domains where the vector $\mathbf{u}$ is continuous together with its derivatives up to second order with respect to the coordinates and time. However, the function $\mathbf{u}(\mathbf{x}, t)$ is continuous (because we do not consider a cracked media) but not smooth enough. For example, the first derivatives of u_3 with respect to the spatial variables are discontinuous on $\partial G(t)$ at the supersonic stage. Thus, we have to provide either a piecewise smooth formulation or a generalised weak formulation of the problem.

If we seek the solution between piecewice regular elastodynamic states on $\mathbb{R}^3 \times [0, t]$, then the sought solution should satisfy some jump conditions:

(i) kinematic (Maxwell) conditions

$$[M_{\alpha i}]^{\pm} = 0,\ M_{\alpha i} = n_{\alpha}^{k}\frac{\partial u_i}{\partial t} + c^k \frac{\partial u_i}{\partial x_{\alpha}}, \tag{13}$$

(ii) dynamic (Sobolev) conditions

$$[M_{4i}]^{\pm} = 0,\ M_{4i} = C_{ijkl}n_{j}^{k}\frac{\partial u_k}{\partial x_l} + \rho c^k \frac{\partial u_i}{\partial t}, \tag{14}$$

where $n_j = \cos(\mathbf{n}, x_i)$ and c^k are the direction and the speed of propagation of the surface $\mathscr{S}_k$. In the region of singular curves of $\mathscr{S}_k$, there are some additional energy restrictions (see for detail, e.g., Brockway, 1972; Borodich, 1990a, 1990c).

3.2. Weak elastodynamic states

We will employ the symbol $L_2(\mathbb{R}_+^3 \times [0, t_*])$ for the class of all tensor-valued functions whose squares are Lebesgue integrable in $\mathbb{R}_+^3 \times [0, t_*]$. If Ω is some domain, then we denote by $C_0^1(\Omega)$ the set of functions continuously differentiable in Ω such that they vanish on a boundary strip (each function has its own strip) of the domain of definition; if Ω is an infinite domain, then we additionally require that functions from $C_0^1(\Omega)$ have compact support. If there exist a vector field $\mathbf{q}$ and rank-*2* tensor field $\mathbf{z}$ that are integrable in $\mathbb{R}_+^3 \times [0, t_*]$ with the following relationships for any $\varphi \in C_0^1(\mathbb{R}_+^3 \times [0, t_*])$

$$\int_0^{t_*}\int_{\mathbb{R}_+^3} \dot{\varphi}_{\alpha} p_{\alpha}\, \mathrm{d}\mathscr{V}\, \mathrm{d}t = -\int_0^{t_*}\int_{\mathbb{R}_+^3} \varphi_{\alpha} q_{\alpha}\, \mathrm{d}\mathscr{V}\, \mathrm{d}t,$$

$$\int_0^{t_*}\int_{\mathbb{R}_+^3} \varphi_{\alpha,\ \beta} p_{\alpha}\, \mathrm{d}\mathscr{V}\, \mathrm{d}t = -\int_0^{t_*}\int_{\mathbb{R}_+^3} \varphi_{\alpha} z_{\alpha\beta}\, \mathrm{d}\mathscr{V}\, \mathrm{d}t,$$

where $\mathrm{d}\mathscr{V}$ denotes $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$, then we say that $\mathbf{p}$ possesses generalised derivatives $\mathbf{q}$ in the Sobolev sense with respect to time and generalised gradient $\mathbf{z}$ on $\mathbb{R}_+^3 \times [0, t_*]$ (see, e.g., Ladyzhenskaya, 1985; Brockway, 1972; Borodich, 1990b, 1990c). We set $\partial_{\beta} p_{\alpha} \equiv z_{\alpha\beta}$ and $\partial p_{\alpha} \equiv q_{\alpha}$.

The ordered pair $[\mathbf{u}, \boldsymbol{\sigma}]$ is a weak dynamic state on $\mathbb{R}_+^3 \times [0, t_*]$ corresponding to the density field ρ and the operator of the constitutive relations $\mathscr{F}$, provided that the following relations are observed (Borodich, 1990c; see also, Brockway, 1972):

$$\mathbf{u} \in W_2^1\left(\mathbb{R}_+^3 \times [0, t_*]\right),$$

$$\sigma_{ij} = \mathscr{F}\left\{\frac{(\partial_k u_l + \partial_l u_k)}{2}\right\} \quad \text{on} \quad \mathbb{R}_+^3 \times [0, t_*]. \tag{15}$$

In addition, for every $\boldsymbol{\varphi} \in C_0^1(\mathbb{R}_+^3 \times [0, t_*])$, the following integral identities are satisfied

$$\int_0^{t_*} \int_{\mathbb{R}_+^3} \left(\rho \dot{\varphi}_\alpha \partial \dot{u}_\alpha - \varphi_{\alpha, j} \sigma_{j\alpha}\right) \mathrm{d}\mathscr{V}\, \mathrm{d}t = 0, \quad \alpha = 1, 2, 3, \tag{16}$$

where W_2^1 is the Sobolev space and C is a space of continuous functions.

We assume that the generalised solution of the dynamic contact problem is among weak elastodynamic states $[\mathbf{u}, \boldsymbol{\sigma}]$ on $\mathbb{R}_+^3 \times [0, t_*]$ and, in addition, the vector $\mathbf{u}$ is continuous but non-smooth, i.e.,

$$\mathbf{u} \in C\left(\mathbb{R}_+^3 \times [0, t_*]\right). \tag{17}$$

Eqs. (15) and (16) are written in the weak formulation of elastodynamic problem instead of Eq. (4). We consider weak states $[\mathbf{u}, \boldsymbol{\sigma}]$ subject to the following generalised initial,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \tag{18}$$

and boundary conditions:

$$u_3(x_1, x_2, 0, t) = g(x_1, x_2, t), \quad (x_1, x_2) \in G(t), \tag{19}$$

$$\int_0^{t_*} \int_{\mathbb{R}_+^3} \left(\rho \dot{\psi}_\alpha \partial \dot{u}_\alpha - \psi_{\alpha, j} \sigma_{\alpha j}\right) \mathrm{d}\mathscr{V}\, \mathrm{d}t + \int_{\mathbb{R}_+^3} v_\alpha^0(\mathbf{x}) \rho(\mathbf{x}) \psi_\alpha(\mathbf{x}, 0)\, \mathrm{d}\mathscr{V} + \int_0^{t_*} \int_{\mathbb{R}^2} T_\alpha \psi_\alpha\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}t = 0, \quad \alpha = 1, 2, 3, \tag{20}$$

instead of Eq. (7).

The last integral identities are satisfied for any functions $\boldsymbol{\psi}$ such that

$$\boldsymbol{\psi} \in C^1\left(\mathbb{R}_+^3 \times [0, t_*]\right), \ \boldsymbol{\psi}(\mathbf{x}, t_*) = 0,$$

$$\psi_3(x_1, x_2, 0, t) = 0 \quad \text{for} \quad (x_1, x_2) \in G(t) \quad \text{and} \quad 0 \le t \le t_*. \tag{21}$$

Here, $\mathbf{u}^0$ and $\mathbf{v}^0$ are the initial displacements and velocities of points of the medium respectively; T_1 and T_2 are the tangential surface tractions given on the whole boundary plane, and T_3 is the normal traction given outside the contact region G; g is a function for the normal displacement which is known on the whole contact region G. It follows from Eq. (7) that these values satisfy the following conditions

$$\mathbf{u}^0(\mathbf{x}) \equiv 0, \quad \mathbf{v}^0(\mathbf{x}) \equiv 0,$$

$$T_\alpha(x_1, x_2, t) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash G(t), \quad \alpha = 1, 2, 3. \tag{22}$$

Thus, Eq. (20) can be written in the following form

$$\int_0^{t_*}\int_{\mathbb{R}_+}\int_{\mathbb{R}^2}(\rho\dot{\psi}_\alpha\partial\dot{u}_\alpha-\psi_{\alpha,j}\sigma_{\alpha j})\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}t+\int_{\mathbb{R}^3_+}0\cdot\rho(\mathbf{x})\psi_\alpha(\mathbf{x},0)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$+\int_0^{t_*}\int_{G(t)}T_\alpha\cdot\psi_\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t=0,\quad \alpha=1,2,3. \tag{23}$$

4. General properties of dynamical contact problems

Let a die contact the boundary of a half-space at the moment $t=0$. and the shape of the die be described by the function f, i.e.,

$$x_3=-f(x_1,x_2).$$

Then we can formulate some general properties of dynamical contact problems which are valid for media with various consitutive relations.

4.1. Supersonic stage of contact

Let us show that, in the problem of vertical pressing, if the initial velocity of a blunt die is non zero, then there exists a time interval on which the speed of propagation of the contact region boundary exceeds the maximum speed of wave propagation in the medium. The proof of the corresponding assertion is obtained by using only geometrical arguments.

Assertion 1. Let the body be smooth and blunt, i.e.,

$$f(x_1,x_2)\in C^1(\mathbb{R}^2),\quad \mathrm{grad}|f(0,0)|=0.$$

Let its velocity be $V(t)$ and $V(0)\neq 0$.
Let the maximum speed (a) of wave propagation in the medium be finite.
Then there exists a time interval $[0, t_a]$, $t_a>0$, on which γ exceeds a.

Proof. Let us denote the cross-section of the body at the height H as $G(H)$ and its boundary as $\partial G(H)$. Then, the vector of the unit normal $\mathbf{n}$ to $\partial G(H)$ is

$$\mathbf{n}=\frac{\mathrm{grad}\,f}{|\mathrm{grad}\,f|}.$$

Let us denote the points of $\partial G(H(t_0))$ as (x_1^*, x_2^*) and the points of $\partial G(H(t_1))$ as (x_1^{**}, x_2^{**}) where $t_1=t_0+\Delta t$. Then we have

$$f(x_1^*,x_2^*)-H(t_0)=0,\quad f(x_1^{**},x_2^{**})-H(t_1)=0.$$

Linearizing the latter equation, we obtain

$$f(x_1^{**},x_2^{**})=f(x_1^*,x_2^*)+\langle\mathrm{grad}\,f(x_1^*,x_2^*),\Delta\mathbf{x}\rangle+o(\Delta\mathbf{x}),\ \Delta\mathbf{x}=(x_1^{**}-x_1^*,x_2^{**}-x_2^*),$$

where $\langle,\rangle$ denotes the scalar product of vectors.

Next, we have

$$H(t_1) = \int_0^{t_1} V(\tau)\mathrm{d}\tau = \int_0^{t_0} V(\tau)\mathrm{d}\tau + V(t_0)\Delta t + o(\Delta t) = H(t_0) + V(t_0)\Delta t + o(\Delta t).$$

Hence

$$\langle \operatorname{grad} f(x_1^*, x_2^*), \Delta \mathbf{x} \rangle = V(t_0)\Delta t$$

or

$$\left\langle \frac{\operatorname{grad} f(x_1^*, x_2^*)}{\left|\operatorname{grad} f(x_1^*, x_2^*)\right|}, \Delta \mathbf{x} \right\rangle = \frac{V(t_0)}{\left|\operatorname{grad} f(x_1^*, x_2^*)\right|}\Delta t.$$

Let us recall that γ is the velocity of the curve ∂G measured along the normal to ∂G

$$\gamma(x_1^*, x_2^*, t_0) = \lim_{\Delta t \to 0} \frac{\langle \mathbf{n}, \Delta \mathbf{x} \rangle}{\Delta t}$$

or

$$\gamma(x_1^*, x_2^*, t_0) = \frac{V(t_0)}{\left|\operatorname{grad} f(x_1^*, x_2^*)\right|}.$$

Since $V(0) > 0$, there exists an interval $[0, T_0]$ when $V(t) > 0$, $t \in [0, T_0]$. Therefore, there exists an interval $[0, t_a]$, $t_a < T_0$, such that

$$\gamma(x_1^*, x_2^*, t_0) > a, \quad t_0 \in [0, t_a], \quad (x_1^*, x_2^*) \in \partial G(H(t_0)) \tag{24}$$

because the body is blunt.

In the case of an isotropic elastic medium, the maximum speed of waves is the speed of propagation of the longitudinal waves a.

As we have mentioned, it is said (see, e.g., Kubenko, 1997) that the process of indentation in the interval $[0, t_a]$ is supersonic.

4.2. Conditions of self-similarity

It is known (see, e.g., Borodich, 1990c) that the similarity in the static Hertz contact problem can be found for bodies of materials whose operators of constitutive relations are homogeneous functions of degree μ with respect to the components of the strain tensor ϵ_{ij}, i.e., for each positive λ we have

$$\mathscr{F}(\lambda\epsilon_{ij}) = \lambda^{\mu}\mathscr{F}(\epsilon_{ij}).$$

Let us consider a hyperelastic medium, i.e., that medium for which a positive defined potential U (the elastic energy) exists. In this case, the constitutive relations have the form

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}}$$

and the stresses σ_{ij} and deformations ϵ_{ij} are independet of the time. The constitutive relations are homogeneous if U is a homogeneous function of degree $\mu+1$ in terms of ϵ_{ij}, i.e.,

$$U(\lambda\epsilon_{ij}) = \lambda^{\mu+1}U(\epsilon_{ij}). \tag{25}$$

The following assertion was proved by Borodich (1990c) for frictionless contact problems. However, it is easy to check in the way used by Borodich (1993) in static contact problems that the assertion is also valid for adhesive or frictional contact problems with Coulomb law of friction, i.e., for problems with Eq. (11) or Eq. (12).

Assertion 2. Let the constitutive relations of the elastic half-space satisfy conditions (Eq. (25)). Let the shape of a convex die be described by a positive, homogeneous function f of degree $d \geq 1$, i.e.,

$$f(x_1, x_2) > 0, \quad \forall (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\}$$

$$f(\lambda x) = \lambda^d f(x), \quad \forall \lambda > 0.$$

Let the die's velocity be $V(t)$ and

$$V(t) = V(1)t^{\frac{(d-1)(\mu+1)}{d+1-\mu(d-1)}}. \tag{26}$$

Assume further that at the moment $t = 1$ the solution of a contact boundary-initial value problem (Eqs. (4), (5), (7), (8), (13) and (14)), and one of the conditions (Eqs. (10)–(12)) is given by $\mathbf{u}(\mathbf{x}, 1)$ and $G(1)$, and the surfaces of discontinuity $\mathscr{S}_k$ are determined by equations $\mathscr{S}_k(\mathbf{x}, 1) = 0$.

Then, the contact boundary-initial value problem at any moment t, $t \in (0, T]$ is satisfied by

$$\mathbf{u}(\mathbf{x}, t) = \lambda^{-d}\mathbf{u}(\lambda\mathbf{x}, 1),$$

$$[(x_1, x_2) \in G(t)] \Leftrightarrow [\lambda x_1, \lambda x_2) \in G(1)],$$

where $\lambda = t^{2/[\mu(d-1)-(d+1)]}$.

The points $\mathbf{x}$ of the surfaces of discontinuity $\mathscr{S}_k$ at the moment t are determined by the equations $\mathscr{S}_k(\lambda\mathbf{x}, 1) = 0$.

Corollary. The resultant force $P(t)$ of contact between the medium and the contacting body is determined by

$$P(t) = P(1)t^{2[2+\mu(d-1)]/[(d+1)-\mu(d-1)]}.$$

Remarks.

1. It follows from Eq. (26) that if $d \neq 1$, then the velocity of the die at the moment $t = 0$ is zero. The speed of the contact region boundary is constant in these self-similar problems. Thus, there is either no supersonic stage of the contact or the extension of the contact region is ever supersonic.
2. The case of linear elastic medium can be obtained by substituting $\mu = 1$ in the above equations. The conditions of the known self-similar problems for this medium:
   (i) a conical die having constant velocity (see, e.g. Kostrov, 1964a, 1964b; Willis, 1973; Bedding and Willis, 1973, 1976; Afanasev and Cherepanov, 1973; Robinson and Thompson, 1974b; Brock, 1976, 1977);

(ii) a uniformly accelerated parabolic die (Afanasev and Cherepanov, 1973) can be obtained by substituting in the above equations $d = 1$ and $d = 2$, respectively.

5. Exact solutions to plane wave problems for anisotropic elastic, viscoelastic, and elastic with initial stresses media

These solutions were obtained by Borodich (1990c, 1998). Here, we write these solution for the sake of completeness of the paper.

5.1. The general solution

Let us assume that the boundary plane $y_3 = 0$ of a homogeneous prestressed elastic half-space is loaded beginning at the time $t = 0$, and the load applied is the same for all points of the plane. Then, we have

$$\mathbf{u}(\mathbf{y}, t) \equiv \mathbf{u}(y_3, t). \tag{27}$$

It follows from Eq. (27) that

$$\frac{\partial \mathbf{u}}{\partial y_1} = \frac{\partial \mathbf{u}}{\partial y_2} = 0. \tag{28}$$

After substituting Eq. (28) into Eq. (A3), the linearized equations of motion become

$$\frac{1}{\rho^*}\omega^*_{3\alpha k3}\frac{\partial^2 u_k}{\partial y_3^2} - \ddot{u}_\alpha = 0. \tag{29}$$

Hence, for both anisotropic elastic and elastic with initial stresses media, we can write the equations of motion as

$$\Lambda_{\alpha k} u_{k,\,33} - \ddot{u}_\alpha = 0, \tag{30}$$

where

$$\Lambda_{\alpha k} = \frac{1}{\rho} C_{\alpha 3k3}$$

and

$$\Lambda_{\alpha k} = \frac{1}{\rho^*}\omega^*_{3\alpha k3},$$

respectively. In the case of an isotropic elastic medium, the matrix $\Lambda_{\alpha k}$ is defined by

$$\Lambda_{\alpha k} = \frac{1}{\rho}\begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda + 2\mu \end{pmatrix}.$$

In the case of linear elasticity, the matrix $(\Lambda_{\alpha k})$ is symmetrical and positive-definite because the tensor C_{ijkl} is symmetric and positive-define. As has been mentioned above, dynamical problems for elastic media with initial stresses are more complex than problems for linear anisotropic elastic media. However, the matrix $(\Lambda_{\alpha k})$ in Eq. (30) is symmetric due to Eqs. (A1) and (9) and positive-definite due to the Hadamard conditions (A7).

It is known (see, e.g., Eringen and Suhubi, 1974; Guz, 1986b) that for real symmetric tensors, the eigenvalues are real and principial directions linked to distinct eigenvalues are mutually orthogonal. Therefore, there are three mutually orthogonal eigenvectors $\mathbf{e}_l$ with real positive eigenvalues C_l^2 such that

$$\mathbf{\Lambda e}_l = C_\alpha^2 \mathbf{e}_\alpha.$$

Let us expand $\mathbf{u}$ in terms of this orthogonal basis $\mathbf{e}_l$

$$\mathbf{u}(y_3, t) = u_1^*(y_3, t)\mathbf{e}_1 + u_2^*(y_3, t)\mathbf{e}_2 + u_3^*(y_3, t)\mathbf{e}_3,$$

where coefficients u_l^* are some scalar functions. Using this orthogonal basis, we obtain that the equations of motion are equivalent to the three plane wave equations

$$C_\alpha^2 \partial^2 u_\alpha^* / \partial y_3^2 - \ddot{u}_\alpha^* = 0.$$

Applying d'Alembert's assertion (see, e.g. Eringen and Suhubi, 1975), we obtain that the general form of a solution to the plane wave problem, for an anisotropic prestressed elastic half-space, is

$$\mathbf{u}(y_3, t) = \Phi_l(y_3/C_l - t)\mathbf{e}_l. \tag{31}$$

Here $\Phi_l(\xi)$ are arbitrary twice continuously differentiable functions of one variable such that $\Phi_l'(0) = \Phi_l''(0) = 0$ and $\Phi_l(\xi) = 0$ for $\xi \geq 0$, where the prime denotes the derivative with respect to the argument of the function.

Thus, the stress vectors, $\mathbf{Q}_3(y_3, t) = Q_{31}\mathbf{i}_1 + Q_{32}\mathbf{i}_2 + Q_{33}\mathbf{i}_3$ and $\boldsymbol{\sigma}_3(x_3, t) = \sigma_{31}\mathbf{i}_1 + \sigma_{32}\mathbf{i}_2 + \sigma_{33}\mathbf{i}_3$, can be represented in the basis $\mathbf{e}_l$ as

$$\mathbf{Q}_3(y_3, t) = \rho^* \mathbf{\Lambda} \partial \mathbf{u}(y_3, t)/\partial y_3 = \rho^* C_l \Phi_l'(y_3/C_l - t)\mathbf{e}_l,$$

$$\boldsymbol{\sigma}_3(x_3, t) = \rho \mathbf{\Lambda u}_{,3}(x_3, t) = \rho c_l \Phi_l'(x_3/c_l - t)\mathbf{e}_l, \tag{32}$$

respectively.

5.2. The boundary-initial value problem for plane waves

Let us now consider the boundary-initial value problem for plane waves. We seek a vector $\mathbf{u}(y_3, t)$ that satisfies the equations of motion, zero initial conditions and the following boundary conditions on the plane $y_3 = 0$:

$$Q_{31}(0, t) = Q_{32}(0, t) = 0, \quad u_3(0, t) = Y(t) \tag{33}$$

where $Y(t)$ is a given twice continuously differentiable function of time.

The former condition in Eq. (33) can be rewritten in the form

$$\mathbf{Q}_3(0, t) = \chi(t)\mathbf{i}_3,$$

where $\chi(t)$ is the desired unknown function.

Let α_l be the coefficients of the expansion of the vector $\mathbf{i}_3$ in the basis $\mathbf{e}_l$, i.e., $\mathbf{i}_3 = \alpha_l \mathbf{e}_l$.

Using the above general solution Eq. (32), we obtain

$$\mathbf{Q}_3(0, t) = \rho^* C_l \Phi_l'(-t)\mathbf{e}_l = \chi(t)\alpha_l \mathbf{e}_l$$

and, therefore, we have

$$\Phi_\beta'(-t) = \alpha_\beta \left(\rho^* C_\beta\right)^{-1} \chi(t).$$

After integration, we obtain

$$\Phi_\beta(-t) = \alpha_\beta \left(\rho^* C_\beta\right)^{-1} \kappa(t), \quad \kappa(t) = -\int_{-t}^{0} \chi(-\tau)\mathrm{d}\tau.$$

On the other hand, we obtain from Eq. (31)

$$u_3(y_3, t) = \langle \mathbf{u}(y_3, t), \mathbf{i}_3 \rangle = \Phi_l(y_3/C_l - t)\alpha_l$$

or

$$u_3(0, t) = Y(t) = \alpha_l \cdot \alpha_l \left(\rho^* C_l\right)^{-1} \kappa(t).$$

Let us define the constant a for an anisotropic elastic medium with initial stresses in accordance with the equation

$$a = \left(\alpha_1^2/C_1 + \alpha_2^2/C_2 + \alpha_3^2/C_3\right)^{-1}, \tag{34}$$

then we obtain

$$\kappa(t) = \rho^* a Y(t).$$

Thus, the exact solution to the problem has the following form

$$\mathbf{u}(y_3, t) = a\alpha_l C_l^{-1} Y(t - y_3/C_l)\mathbf{e}_l \tag{35}$$

and the stress vectors $\boldsymbol{\sigma}_3(x_3, t)$ and $\mathbf{Q}_3(y_3, t)$ have the following form

$$\boldsymbol{\sigma}_3(x_3, t) = -\rho a \alpha_l Y'(t - x_3/C_l)\mathbf{e}_l, \quad \mathbf{Q}_3(y_3, t) = -\rho^* a \alpha_l Y'(t - y_3/C_l)\mathbf{e}_l. \tag{36}$$

5.3. Plane waves in orthotropic viscoelastic media

Let us now consider the boundary-initial value problem for plane waves in orthotropic viscoelastic media. The constitutive equations Eq. (5) for such media can be written as (see, e.g., Rabotnov, 1980)

$$\sigma_{ij} = \tilde{C}_{ijkl} \partial u_k / \partial x_l, \quad \tilde{C}_{ijkl} = C_{ijkl} - \Gamma_{ijkl}^*,$$

where C_{ijkl} is the tensor of instantaneous elastic modulus of the material and Γ_{ijkl}^* is the integral Volterra operator with a differnce kernel.

Again, we seek a vector $\mathbf{u}(x_3, t)$ that satisfies the equations of motion, zero initial conditions and the boundary conditions (33) on the plane $x_3 = 0$:

$$\sigma_{31}(0, t) = \sigma_{32}(0, t) = 0, \quad u_3(0, t) = Y(t).$$

The type of these boundary conditions does not change throughout the period of the problem under consideration. Therefore, the Volterra principle is valid for the problem. Applying the principle, we can take the solution Eq. (36) and replace the function of elastic constants a by the corresponding operator $\{a+A^*\}$, where A^* is the Volterra operator with the kernel $A(t)$. Note that for orthotropic media, we have $\mathbf{e}_3=\mathbf{i}_3$, $\alpha_1=\alpha_2=0$, $\alpha_3=1$ and, hence, it follows from Eq. (34) that $a=(C_{3333}/\rho)^{1/2}$. Then we have

$$\boldsymbol{\sigma}_3(0,\ t) = -\rho\tilde{A}\alpha_l Y'(t)\mathbf{e}_l = -\rho\left[aY'(t) + \int_0^t Y'(t-\tau)A(\tau)\,\mathrm{d}\tau\right]\mathbf{i}_3 =$$

$$-\rho\left[aY'(t) - Y(0)A(t) + Y(t)A(0) + \int_0^t Y(t-\tau)A'(\tau)\,\mathrm{d}\tau\right]\mathbf{i}_3.$$

Taking into account that $Y(0)=0$, we obtain

$$\boldsymbol{\sigma}_3(0,\ t) = -\rho\left[aY'(t) + Y(t)A(0) + \int_0^t Y(t-\tau)A'(\tau)\mathrm{d}\tau\right]\mathbf{i}_3. \tag{37}$$

6. Method of integral characteristics for weak elastodynamic states

For a weak elastodynamic state (Eqs. (15) and (16)), we introduce the following integral characteristics

$$\mathbf{w}(x_3,\ t) = \iint_{\mathbb{R}^2} \mathbf{u}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

$$\boldsymbol{\lambda}_i(x_3,\ t) = \iint_{\mathbb{R}^2} \partial_i\mathbf{u}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

$$\boldsymbol{\lambda}_4(x_3,\ t) = \iint_{\mathbb{R}^2} \partial\dot{\mathbf{u}}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

$$\Sigma_{ij}(x_3,\ t) = \iint_{\mathbb{R}^2} \sigma_{ij}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{38}$$

6.1. General properties of the integral characteristics

It was shown by Borodich (1990b, 1990c) that for the integral characteristics Eq. (38), the following lemma is valid.

Lemma. Let the support of the functions $\mathbf{u}$ and σ_{ij} be spatially bounded for $0 \le t \le t_*$. Then the integral characteristics introduced in Eq. (38) possess the following properties:

1. $\mathbf{w}$, $\boldsymbol{\lambda}_i$, $\boldsymbol{\lambda}_4$ are determined almost everywhere in $\mathbb{R}_+ \times [0,\ t_*]$ and are summable;
2. $\mathbf{w}(x_3,\ t) \in W_2^1(\mathbb{R}_+ \times [0,\ t_*])$;
3. $\boldsymbol{\lambda}_3$ and $\boldsymbol{\lambda}_4$ are generalised derivatives of the function $\mathbf{w}(x_3,\ t)$ with respect to x_3 and t, respectively;

4. $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ equal zero for almost all x_3 and t;
5. $\boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4 \in L_2(\mathbb{R}_+ \times [0, t_*])$.

We suppose that a weak elastodynamic state which gives a solution to the boundary-initial value problem (Eqs. (15)–(22)) exists and is unique.

The discussions of the questions of the uniqueness of weak solutions to hyperbolic equations, as well as to the first and the second boundary-initial value problems in linear elastodynamics, can be found in Ladyzhenskaya (1985) and Brockway (1972). In addition, general uniqueness and existence questions for problem of dynamical indentation were considered by Georgiadis and Barber (1993).

Due to the boundedness of the supports of the functions $\mathbf{u}$ and σ_{ij} at the initial time and the finiteness of the maximal speed of wave propagation in the medium, the supports of these functions are bounded for any finite t. Thus, the conditions of the above lemma are satisfied.

In this paper, we consider the frictional contact problems only for dies which can be represented as a blunt convex rigid body having an arbitrary shape with two orthogonal planes of symmetry $0x_1x_3$ and $0x_2x_3$, which are both orthogonal to the boundary of half-space. These planes are planes of symmetry for the problem (Eqs. (15)–(22)) under consideration.

As a consequence, we obtain that for any kind of boundary conditions, frictional, adhesive or frictionless, the following conditions of symmetry are satisfied

$$\sigma_{31}(x_1, x_2, 0, t) = -\sigma_{31}(x_1, -x_2, 0, t), \quad \sigma_{32}(x_1, x_2, 0, t) = -\sigma_{32}(-x_1, x_2, 0, t)$$

for any point of the boundary surface. Then, for any conditions on the tangential surface tractions T_1 and T_2, we have

$$\Sigma_{31}(0, t) = \Sigma_{32}(0, t) = 0, \quad t \in [0, t_*]. \tag{39}$$

Using the lemma and Eq. (39), we obtain from Eq. (23) the following equations for integral characteristics $\mathbf{w}$ and σ_{ij}

$$\int_0^{t_*} \int_{\mathbb{R}_+} (\rho \dot{\theta}_\alpha \partial \dot{w}_\alpha - \theta_{\alpha,3} \Sigma_{3\alpha})\, \mathrm{d}x_3\, \mathrm{d}t = 0. \tag{40}$$

The last identities are satisfied for any functions $\boldsymbol{\theta}$, such that

$$\boldsymbol{\theta}(x_3, t) \in C^1(\mathbb{R}_+ \times [0, t_*]),$$

$$\boldsymbol{\theta}(x_3, t_*) = 0,$$

$$\boldsymbol{\theta}(0, t) = 0, \quad 0 \le t \le t_*.$$

Thus, from Eqs. (15), (18), (19), (22) and (40), we obtain that the integral characteristics $\mathbf{w}$ and σ_{ij} of the problem under consideration, introduced in Eq. (38) satisfy the following boundary-initial value problem

$$\mathbf{w} \in C(\mathbb{R}_+ \times [0, t_*]); \quad \mathbf{w} \in W_2^1(\mathbb{R}_+ \times [0, t_*]),$$

$$\Sigma_{ij} = \mathscr{F}\{(\partial_3 w_k + \partial_3 w_l)/2\} \quad \text{on} \quad \mathbb{R}_+ \times [0, t_*],$$

$$\mathbf{w}(\mathbf{x}, 0) = 0,$$

$$w_3(0, t) = Y(t),$$

$$\int_0^{t_*} \int_{\mathbb{R}_+} (\rho \dot{\theta}_\alpha \partial \dot{w}_\alpha - \theta_{\alpha,3} \Sigma_{3\alpha}) \, \mathrm{d}x_3 \, \mathrm{d}t = 0, \tag{41}$$

where $Y(t)$ is a known function, given by

$$Y(t) = \int \int_{\mathbb{R}^2} u_3(x_1, x_2, 0, t) \, \mathrm{d}x_1 \, \mathrm{d}x_2. \tag{42}$$

The problem (41) for the integral characteristics is a problem concerning plane wave propagation in the same linear medium.

6.2. Weak solutions to the plane wave problems

Let us consider a homogeneously prestressed elastic medium. As in the above three-dimensional dynamic problem (Eqs. (15)–(17), (19) and (21)–(23)), we seek a vector $\mathbf{u}$ such that

$$\mathbf{u} \in W_2^1(\mathbb{R}_+ \times [0, t_*]); \quad \mathbf{u} \in C(\mathbb{R}_+ \times [0, t_*]); \quad \mathbf{u}(y_3, 0) = 0$$

$$u_3(0, t) = Y(t), \quad \int_0^{t_*} \int_{\mathbb{R}_+} (\dot{\theta}_\beta \dot{\partial} u_\beta - \theta_{\beta,3} Q_{3\beta}) \, \mathrm{d}y_3 \, \mathrm{d}t = 0. \tag{43}$$

The integral identities are satisfied for any functions $\boldsymbol{\theta}$ such that

$$\boldsymbol{\theta} \in C^1(\mathbb{R}_+ \times [0, t_*]), \quad \boldsymbol{\theta}(y_3, t_*) = 0,$$

$$\theta_3(y_3, t) \in C_0^1(\mathbb{R}_+) \quad \text{for} \quad 0 \le t \le t_*.$$

Assertion 3. The generalized solution to the above weak formulation of the plane wave problem Eq. (43) has the form of Eqs. (35) and (36).

Indeed, if $Y(t)$ is a continuous function of one argument, then Eqs. (35) and (36) satisfy Eq. (43), i.e., it is a weak solution of the plane wave problem. It follows from the uniqueness theorem that this solution is unique.

We see that the weak formulation of the above plane wave problem Eq. (43) and the one dimensional problem Eq. (41) for the integral characteristics are the same. Thus, we can formulate the following assertion.

Assertion 4. The weak solution to the one-dimensional dynamical problem Eq. (41) has the form

$$\mathbf{w}(y_3, t) = a\alpha_l Y(t - y_3/C_l)\mathbf{e}_l/C_l. \tag{44}$$

Corollary. The stress vector $\boldsymbol{\Sigma} = \{\Sigma_{3i}\}$ has the form

$$\boldsymbol{\Sigma}(y_3, t) = -\rho^* \alpha_l Y'(t - y_3/C_l)\mathbf{e}_l. \tag{45}$$

where the prime denotes the generalized derivative in the Sobolev sense. Note that ρ^* should be replaced by ρ in the case of linear elasticity.

During the first, supersonic stage of contact, we obtain from Eqs. (8), (19), (42) and (45)

$$Y(t) = \int_{\mathbb{R}^2} g(y_1, y_2, t)\, \mathrm{d}y_1\, \mathrm{d}y_2, \quad Y'(t) = V(t)S(t),$$

$$\boldsymbol{\Sigma}(y_3, t) = -\rho a V(t - y_3/C_3)S(t - y_3/C_3)\mathbf{e}_3. \tag{46}$$

Note that $P(t) = -(\boldsymbol{\Sigma}(0, t), \mathbf{i}_3)$. Thus, if the die is a blunt convex rigid body having an arbitrary shape with two orthogonal planes of symmetry, which are both orthogonal to the boundary of half-space, then for any kind of boundary conditions frictional, adhesive or frictionless, we obtain from Eq. (46) that the instantaneous value of the force required to indent the die during the supersonic stage of contact is defined by Eq. (2).

7. Integral characteristics for problems of pressing with rotation of body axes

Let us consider frictionless problems of pressing a die into a continuous half-space in the case when the axes of the die are rotated. It is supposed in this problem that both the velocity $\mathbf{v}^K(t)$ of the centre of mass of the body K, and its angular velocity $\boldsymbol{\omega}$ are known.

On the supersonic stage of propagation of the boundary of the contact region, the following relationships were obtained for the resulting force P of contact between the medium and the contacting body and the resulting moments M_i of contacting stresses about axes x_1 and x_2

$$P = \rho a\left[v_3^K S + \omega_1 S_1^* - \omega_2 S_2^*\right] \tag{47}$$

and

$$M_1 = \rho a\left[v_3^K S_1^* + \omega_1 I_{22}^* - \omega_2 I_{12}^*\right]; \quad M_2 = -\rho a\left[v_3^K S_2^* + \omega_1 I_{12}^* - \omega_2 I_{11}^*\right], \tag{48}$$

where $Ox_1x_2x_3$ is the initial coordinate system, O is the point of initial contact between the body and the half-space taken as the coordinate origin, $O_1x_1'x_2'x_3'$ is the coordinate system with axes parallel to the initial system, O_1 is the projection of the point K onto the boundary plane, S_i^* and I_{ij}^* are the static moment and the moment of inertia of the contact region G about the axis x_i', respectively (Fig. 2).

In the case of an isotropic elastic medium, Eqs. (47) and (48) were announced by Gorshkov and Tarlakovskii (1987). However, a detailed proof was not available for a rather long time after this. The author used his method of integral characteristics and obtained that Eq. (47) is an elementary consequence of Eq. (45) and, therefore, it is valid in the case of an arbitrary anisotropic elastic medium with constant a from Eq. (34) (Borodich, 1990c, 1991). Indeed, on the supersonic stage, the integral of vertical displacements of points of the boundary plane is equal to the integral of vertical displacements of points of the contact region, this integral in turn is equal to the volume of indented part of the body. However, the velocities of these points $v_3(x_1, x_2, 0, t)$ in the region $G(t)$ are not the same. Using the Euler theorem for velocities of body points, we obtain

$$v_3(x_1', x_2', 0, t) = v_3^K(0, 0, 0, t) + \omega_1 x_2' - \omega_2 x_1'.$$

Substituting this formula in the expression for $Y'(t)$, we obtain

$$Y'(t) = \int_{G(t)} v_3(x_1', x_2', 0, t)\, dx_1'\, dx_2' = v_3^K S + \omega_1 S_1^* - \omega_2 S_2^*.$$

Then it was shown that Eq. (48) are valid for an anisotropic elastic medium when the planes Ox_1x_3 and Ox_2x_3 are planes of symmetry, i.e., for an orthotropic medium. To prove this result, two auxiliary problems were solved, namely

1. a self-similar problem of finding a resulting force $\Sigma_{33}^h(0, t; x_1^0, x_2^0)$ and moments M_1^h, M_2^h, arose in the medium caused by a concentrated velocity or displacement applied at the point (x_1^0, x_2^0);
2. superposition of obtained solutions.

Such a superposition is possible because the problem is linear.

Let us show that Eqs. (47) and (48) are valid for an elastic medium with initial stresses when the planes Oy_1y_3 and Oy_2y_3 are planes of symmetry of the problem.

First, we consider the auxiliary problem concerning a concentrated velocity. The boundary conditions are the following

$$u_3^*(y_1, y_2, 0, t) = h(t)\delta(y_1 - y_1^0, y_2 - y_2^0) \quad \text{or} \quad v_3^*(y_1, y_2, 0, t) = \mathscr{H}(t)\delta(y_1 - y_1^0, y_2 - y_2^0),$$

$$Q_{31}(y_1, y_2, 0, t) = Q_{32}(y_1, y_2, 0, t) = 0.$$

In this paragraph, we denote by $\mathscr{H}$ the Heaviside step function and $h'(t) \equiv \mathscr{H}(t)$.

The displacements at all points of the boundary plane with the exception of y_1^0 and y_2^0 are zero. Therefore, there are some external stresses $Q_{33}^h(y_1, y_2, 0, t)$, which hold the boundary plane $y_3 = 0$.

It follows from the boundary conditions that $Y(t) = h(t)$ and $Y'(t) = \mathscr{H}(t)$. Using the solution for the plane wave problem, we obtain

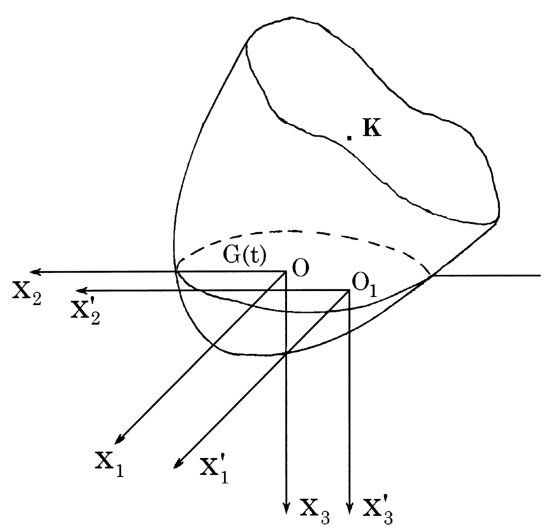

Fig. 2. Coordinate systems in the problem of pressing with rotation of the die axes (after Borodich, 1990c).

$$\Sigma_{33}^h\big(0,\ t;\ y_1^0,\ y_2^0\big) = -\rho^* a h'(t), \quad a = \sqrt{\omega_{3333}^*/\rho^*}.$$

The stresses result in moments M_1^h and M_2^h about axes y_1 and y_2, respectively. Due to the symmetry of the considered problem, we have

$$Q_{33}^h\big(y_1^0 + \Delta y_1,\ y_2,\ 0,\ t\big) = Q_{33}^h\big(y_1^0 - \Delta y_1,\ y_2,\ 0,\ t\big)$$

for any Δy_1. Therefore, we have

$$Q_{33}^h\big(y_1^0 + \Delta y_1,\ y_2,\ 0,\ t\big)\big(y_1^0 + \Delta y_1\big) + Q_{33}^h\big(y_1^0 - \Delta y_1,\ y_2,\ 0,\ t\big)\big(y_1^0 - \Delta y_1\big)$$
$$= y_1^0\Big[Q_{33}^h\big(y_1^0 + \Delta y_1,\ y_2,\ 0,\ t\big) + Q_{33}^h\big(y_1^0 - \Delta y_1,\ y_2,\ 0,\ t\big)\Big].$$

Substituting this formula in the definition of the moment M_2^h, we obtain

$$M_2^h\big(t;\ y_1^0,\ y_2^0\big) = \iint_{\mathbb{R}^2} Q_{33}^h(\ y_1,\ y_2,\ 0,\ t) y_1\ \mathrm{d}y_1\ \mathrm{d}y_2 = y_1^0 \Sigma_{33}^h\big(0,\ t;\ y_1^0,\ y_2^0\big) = -y_1^0 \rho^* a \mathscr{H}(t).$$

In a similar way, we obtain

$$M_1^h\big(t;\ y_1^0,\ y_2^0\big) = -\iint_{\mathbb{R}^2} Q_{33}^h(\ y_1,\ y_2,\ 0,\ t) y_2\ \mathrm{d}y_1\ \mathrm{d}y_2 = -y_2^0 \Sigma_{33}^h\big(0,\ t;\ y_1^0,\ y_2^0\big) = y_2^0 \rho a^* \mathscr{H}(t).$$

This is the solution of the auxiliary problem concerning a concentrated velocity.

Let us now consider the initial problem in the coordinate system $O_1 y_1' y_2' y_3'$ which is introduced in a similar way as the system $O_1 x_1' x_2' x_3'$ was introduced. The instantaneous value of the velocity at a point $(y_1'^{\,0},\ y_2'^{\,0})$ of the contact region is

$$v_3\big(y_1'^{\,0},\ y_2'^{\,0},\ 0,\ t\big) = v_3^K(0,\ 0,\ 0,\ t) + \omega_1 y_2'^{\,0} - \omega_2 y_1'^{\,0}.$$

Let us remember that we consider a linearized problem. Hence, the velocity causes the following moments about the axes $O' y_1'$ and $O' y_2'$

$$M_1\big(t;\ y_1'^{\,0},\ y_2'^{\,0}\big) = y_2'^{\,0} \rho^* a v_3\big(y_1'^{\,0},\ y_2'^{\,0},\ 0,\ t\big), \quad M_2\big(t;\ y_1'^{\,0},\ y_2'^{\,0}\big) = -y_1'^{\,0} \rho * a v_3\big(y_1'^{\,0},\ y_2'^{\,0},\ 0,\ t\big).$$

Using superposition of the solutions obtained, we get

$$M_2(t) = \iint_{G(t)} M_2\big(t;\ y_1'^{\,0},\ y_2'^{\,0}\big)\ \mathrm{d}y_1'^{\,0}\ \mathrm{d}y_2'^{\,0} =$$
$$-\rho^* a \iint_{G(t)} y_1'^{\,0}\big[v_3^K(0,\ 0,\ 0,\ t) + \omega_1 y_2'^{\,0} - \omega_2 y_1'^{\,0}\big]\ \mathrm{d}y_1'^{\,0}\ \mathrm{d}y_2'^{\,0}.$$

This leads to Eq. (48), where ρ should be replaced by ρ^*.

8. Exact solutions to problems of collision

The papers concerning impact problems often assumed that the velocity of the body is constant during the supersonic stage of vertical impact. (see, for example, Skalak and Feit, 1966; Kubenko and Popov, 1989). However, in the problem of vertical collision, the velocity of the body is determined from Newton's equation

$$m\dot{V}(t) = -P(t) \tag{49}$$

where m is the mass of the die.

Only one solution was known to the collision problem with the use of Eq. (49). This was the problem of frictionless impact on an isotropic elastic half-space (Thompson and Robinson, 1977). Then the solutions were obtained for frictionless impact on an acoustic half-space (Borodich, 1988b) and an anisotropic elastic half-space (Borodich, 1990b). Finally, solutions were obtained for problems of frictional impact on an isotropic elastic half-space (Borodich and Gomatam, 1998).

Now solutions will be obtained for problems of frictional impact on anisotropic elastic and elastic with initial stresses media. We will obtain expressions for the relationships between time, depth of indentation and velocity of the body. In particular case, when the body is a elliptic paraboloid or an elliptic cone, the expressions have simple algebraic forms.

8.1. General solution

If the body has two orthogonal planes of symmetry $0x_1x_3$ and $0x_2x_3$ ($0y_1y_3$ and $0y_2y_3$, respectively) which are both orthogonal to the boundary of half-space then we obtain from Eqs. (49) and (2)

$$m\dot{V}(t) = -\rho a V(t) S(t). \tag{50}$$

Let us denote $v[H(t)] \equiv V(t)$. Taking into account that $\dot{H} = V$ (see Eq. (9)), we have

$$\dot{V}(t) = V\frac{\mathrm{d}v}{\mathrm{d}H}. \tag{51}$$

Substituting Eq. (51) into Eq. (50), we obtain

$$m\frac{\mathrm{d}v}{\mathrm{d}H} = -\rho a s(H),$$

where

$$s[H(t)] \equiv S(t). \tag{52}$$

By integrating Eq. (52), we obtain the velocity of the die as a function of H

$$v(H) = V(0) - \rho\frac{a}{m}Y(H),$$

where

$$Y(H) = \int_0^H s(h)\mathrm{d}h \tag{53}$$

and the relationship between time and depth of penetration to be

$$t = \int_0^H \frac{\mathrm{d}h}{V(0) - \rho a m^{-1} Y(h)}. \tag{54}$$

It is easy to see that $Y(H)$ is the volume of the die under the section at height H. Eqs. (53) and (54) give the exact solution of the problem considered.

8.2. Impact of an elliptic paraboloid

Let us consider a particular case, when the body is described by a quadratic form

$$f(x_1, x_2) = B'_1 x_1^2 + B'_{12} x_1 x_2 + B'_2 x_2^2.$$

By rotation of the coordinate axes, the form can be transformed into

$$f\left(x_1^*, x_2^*\right) = B_1 x_1^{*2} + B_2 x_2^{*2}, \quad B_2 \geq B_1 > 0,$$

i.e., we have obtained an elliptic paraboloid which is a figure with two orthogonal planes of symmetry $0x*_1 x_3$ and $0x*_2 x_3$, which are both orthogonal to the boundary of the half-space.

In this case, the exact solution of the impact problem (53) and (54) can be written in simple algebraic functions

$$s(h) = \frac{\pi h}{\sqrt{B_1 B_2}}, \quad Y(h) = \frac{\pi h^2}{2\sqrt{B_1 B_2}},$$

$$t = \frac{1}{K} \ln\left|\frac{1 + HV^{-1}(0)K/2}{1 - HV^{-1}(0)K/2}\right|, \quad K = \sqrt{\frac{2\pi\rho V(0)a}{m\sqrt{B_1 B_2}}}.$$

We can resolve the problem and obtain the following direct relationships between time t and desired functions

$$H(t) = \frac{2V(0)}{K} \frac{e^{Kt} - 1}{e^{Kt} + 1},$$

$$V(t) = 4V(0)\frac{e^{Kt}}{(e^{Kt} + 1)^2}, \quad P(t) = -4V(0)mK\frac{e^{Kt} - e^{2Kt}}{(e^{Kt} + 1)^3}. \tag{55}$$

Note the solution to the frictionless problem presented by Thompson and Robinson (1977) had some misprints and the constants of the solution were wrong.

8.3. Impact of a blunt four-sided pyramid

Let us consider another particular case, namely the case when the body is described by a blunt four-sided pyramid

$$f(x_1, x_2) = B_1|x_1| + B_2|x_2|, \quad B_2 \geq B_1 > 0.$$

We assume that the pyramid is blunt enough to write all boundary conditions on the surface $x_3 = 0$. In this case, the exact solution of the impact problem (Eqs. (52) and (53)) is given by the following formulae

$$s(h) = \frac{2h^2}{B_1 B_2}, \quad Y(h) = \frac{2h^3}{3B_1 B_2}$$

$$t = \int_0^H \frac{\mathrm{d}h}{V(0) - 2\rho a(3mB_1B_2)^{-1}h^3} = \frac{1}{V(0)}\int_0^H \frac{\mathrm{d}h}{1 - h^3/L_1^3},$$

where

$$L_1 = \left(\frac{3mB_1B_2V(0)}{2\rho a}\right)^{1/3}.$$

The last integral can be written in the following form (use Eq. 2.126 in Gradshteyn and Ryzhik, 1965)

$$t = \frac{1}{V(0)}\frac{L_1}{3}\left[-\frac{1}{2}\ln\frac{(H-L_1)^2}{H^2 + L_1H + L_1^2} + \sqrt{3}\arctan\frac{H\sqrt{3}}{2L_1 + H}\right]. \tag{56}$$

8.4. Impact of an elliptic cone

Next, consider the case when the body is described by a blunt elliptic cone

$$f(x_1, x_2) = \sqrt{B_1x_1^2 + B_2x_2^2}, \quad B_2 \ge B_1 > 0.$$

We suppose again that the cone is blunt enough to write all boundary conditions on the surface $x_3 = 0$. The exact solution of the impact problem (Eqs. (52) and (53)) is given by the following formulae

$$s(h) = \frac{\pi h^2}{\sqrt{B_1B_2}},$$

$$Y(h) = \frac{\pi h^3}{3\sqrt{B_1B_2}}$$

and

$$t = \int_0^H \frac{\mathrm{d}h}{V(0) - \rho a\pi(3m\sqrt{B_1B_2})^{-1}h^3} = \frac{1}{V(0)}\int_0^H \frac{\mathrm{d}h}{1 - h^3/K_1^3},$$

where

$$K_1 = \left(\frac{3m\sqrt{B_1B_2}V(0)}{\rho a\pi}\right)^{1/3}.$$

As above, the last integral can be written in the following form

$$t = \frac{1}{V(0)}\frac{K_1}{3}\left[-\frac{1}{2}\ln\frac{(H-K_1)^2}{H^2 + K_1H + K_1^2} + \sqrt{3}\arctan\frac{H\sqrt{3}}{2K_1 + H}\right]. \tag{57}$$

Note that since pyramids and cones are not smooth, Eq. (24) does not hold for all $V(0)$. We have that max|grad f(x_1, x_2)| is equal to $(B_1^2 + B_2^2)^{1/2}$ and $B_2^{1/2}$ for pyramid and cone respectively. Therefore, we obtain that $\min\gamma(x_1, x_2, t) \le V(0)/\max|\mathrm{grad} f(x_1, x_2)|$. Thus, if

$$V(0) \leq a \max|\text{grad} f(x_1, x_2)|,$$

there is no supersonic stage of contact and Eqs. (56) and (57) are not valid.

9. Discussion and conclusion

For both anisotropic elastic and elastic with initial stresses media, we have obtained solutions for impact problems which were solved earlier for only an isotropic linear elastic medium (Thompson and Robinson, 1977; Borodich and Gomatam, 1998). Exact expressions have been derived for the relationships between time, depth of indentation, velocity of the body and contact force. If the body is an elliptic paraboloid, a blunt four-sided pyramid or an elliptic cone, some of these formulae are expressible in terms of elementary functions. The expressions obtained are independent of the boundary conditions prevailing in the contact region.

We have seen that the method of integral characteristics of solutions to boundary-initial value contact problems (Borodich, 1990a, 1990b, 1990c) is very effective. It allows us to find the resultant contact force and the resulting moments M_i of contacting stresses in the problem of frictionless pressing with rotation of the die axes.

We have also found the resultant contact force in the problem of both frictional and frictionless pressing normal to the boundary plane of the anisotropic continuous half-spaces. Note that expressions for resultant forces are not always as simple as Eq. (2). Indeed, it follows from Eq. (37) that the expression in the case of vertical pressing into a linear viscoelastic orthotropic medium has the following form

$$P(t) = \rho\left[aY'(t) + Y(t)A(0) + \int_0^t Y(t-\tau)A'(\tau)\mathrm{d}\tau\right].$$

The exact solutions to impact problems can be used to estimate the accuracy of the used numerical methods. For example, Eq. (2) was used by Iarve (1989) (see also Bogdanovich and Iarve, 1992) for such estimations. Evidently, Eq. (55) is more useful for such a purpose.

It should be noted that the used method is valid for media with linear constitutive relations only when it is known that nonlinear plastic effects are often quite significant for the problem. Another restriction is that Eq. (2), is valid only on the short-term supersonic stage of contact, while Eq. (3) cannot usually be used in an effective way because the mean velocity $V_1(t)$ is unknown. Nevertheless, Eq. (3) can be used to estimate the energy dissipating by waves during the contact between a die and a half-space.

To take into account the plastic effects, an approach was suggested by Borodich (1992) (see also Antonov and Borodich, 1993; Borodich and Goldstein, 1995). The idea of the approach goes back to Simonov and Flitman (1966). This approach combines the similarity consideration of the quasi-static Hertz impact problem for non-linear media (Borodich, 1990c, 1993) and the results concerning the resultant force in the dynamic impact problem. Thus, the above results concerning integral characteristics can be used to estimate the energy dissipation during elastic–plastic collission. We intend to continue our studies in this direction.

Acknowledgements

Thanks are due to the Centre de Recerca Matemàtica, Institut d'Estudis Catalans (Bellaterra, Barcelona) for funding a visit during which this work was completed. The author is also grateful to Mr. C.A. MacDonald (Glasgow Caledonian University) for technical help.

Appendix A. Linearized equations for homogeneously prestressed elastic media

The equations of the theory of nonlinear elasticity may be written using a variety of stress and strain tensors and linearized equations for bodies with initial stresses may be written in a variety of forms. Information concerning dynamical problems for homogeneously deformed nonlinear elastic solids can be found elsewhere (see, e.g., Eringen and Suhubi, 1974; Guz, 1986aGuz, 1986b). Here, we recall only the formulations which are necessary to study the boundary-initial value problems for hyperelastic Hadamard materials with finite initial deformations.

Consider a nonlinear elastic orthotropic medium occupying a half-space. Let this medium be prestressed. Let us denote all quantities pertaining to the initial static stress state by superscript '(A)'. We will call the state of the body A. Suppose that the wave motion results in a small perturbation of the initial stress-state in the half-space. Considering linearized problems with initial stresses, we will denote quantities pertaining to the underformed state by a superscript '(0)', and the small perturbations will be used without any subscripts or superscripts. Thus, for the vector of displacements in the underformed state $\mathbf{u}^{(0)}$, the following expression can be written, $\mathbf{u}^{(0)} = \mathbf{u}^{(A)} + \mathbf{u}$.

Let us introduce the Lagrangian coordinate system $OX^1X^2X^3$, which coincides with the Cartesian coordinate system in the underformed or *natural* state. It is known that covariant and contravariant components coincide with each other in the Cartesian coordinates and we can write $X_k \equiv X^k$. Let $\mathbf{I}_k$ be unit vectors lying along the coordinates X_k, i.e., they are rectangular base vectors. We orient the X_3 axis into the interior of the half-space $\mathbb{R}^3_+$ and the X_1 and X_2 axes along its boundary. Let us suppose that axes of elastic symmetry coincide with the X_1, X_2, X_3 axes.

Let us introduce another system of the Lagrangian coordinates y_1, y_2, y_3 which in the *initial* deformed state A coincide with Cartesian coordinates. Let $\mathbf{i}_k$ be unit vectors lying along the coordinates y_k.

The initial stress field is assumed to be constant for all points of the half-space, i.e., the field is homogeneous. Then the extension ratios λ_i of material fibres directed along x_i axes are constant and for the coordinates of a material particle in the natural state and state A, we have $y_i = \lambda_\alpha X_\alpha$. If δ_{ij} is the Kronecker delta, then the following expression can be written

$$u_j^{(A)} = \delta_{\alpha j}(\lambda_\alpha - 1)X_\alpha = \delta_{\alpha j} y_\alpha (\lambda_\alpha - 1)/\lambda_\alpha,$$

due to the homogeneity of the initial stresses.

Let us write equations in the domains where the vector $\mathbf{u}$ is continuous, together with its derivatives up to second order with respect to the coordinates and time.

In the system X_i, we will use the non-symmetric increment in the Kirchhoff (Piola–Kirchhoff) tensor of stresses, which will be denoted by t_{mn}. The linearized constitutive relations for homogeneously deformed elastic solids can be written as (see, e.g. Guz, 1986a)

$$t_{ij} = \omega_{ijkl} \frac{\partial u_k}{\partial X_l},$$

where components of the tensor ω_{ijkl} are independent of the coordinates of the point due to homogeneity of the initial stresses. However, the components ω_{ijkl} depend on the initial deformations and the type of energy function which is used to describe the elastic behaviour of the material.

It is known (see, e.g. Guz, 1986a) that in the general case

$$\omega_{ijkl} = \omega_{lkji}, \quad \omega_{ijkl} \neq \omega_{ijlk}, \quad \omega_{ijkl} \neq \omega_{klij}. \tag{A1}$$

The tensor C_{ijkl} has larger symmetry properties Eq. (6). Hence, dynamical problems for elastic with initial stresses media are more complex than for anisotropic elastic media and the above-mentioned methods cannot be applied in a direct way to solve dynamical problems for elastic bodies with initial stresses. This is one of the reasons that dynamical contact problems for prestressed solids were considered in a few papers only (see, e.g. Babich, 1987; Borodich, 1991).

The equations of motion have the following form

$$\omega_{ijkl}\frac{\partial^2 u_k}{\partial X_i \partial X_l} - \rho^{(0)}\frac{\partial^2 u_j}{\partial t^2} = 0, \tag{A2}$$

where $\rho^{(0)}$ is the density of the material in the natural state.

Now let us write all relations of the linearized theory of elasticity for a medium with initial stresses using coordinates y_i and refer all quantities to the surface elements in the initial deformed state A. Let $Q_{ij}(\mathbf{y}, t)$ be components in the Cartesian system of the non-symmetric stress tensor $\mathbf{Q}$, linearly related to the components of the Kirchhoff stress tensor. The components of $\mathbf{Q}$ are related to the area units in the state A. The component Q_{nl} is the lth component of the stress vector $\mathbf{Q}_n$ acting on the positive side of the coordinate surface with the nth component of the normal, i.e., $\mathbf{Q}_n = Q_{nl}\mathbf{i}_l$.

For homogeneously deformed elastic solids, the linearized constitutive relations and the equations of motion can be written as

$$Q_{ij} = \omega^*_{ijkl}\frac{\partial u_k}{\partial y_l}$$

and

$$\omega^*_{ijkl}\frac{\partial^2 u_k}{\partial y_i y_l} - \rho^*\frac{\partial^2 u_j}{\partial t^2} = 0, \tag{A3}$$

where

$$\omega^*_{\alpha jk\beta} = \lambda_\alpha\lambda_\beta\lambda_1\lambda_2\lambda_3\omega_{\alpha jk\beta}, \quad \rho^* = \frac{1}{\lambda_1\lambda_2\lambda_3}\rho^{(0)}. \tag{A4}$$

From Eqs. (A1) and (A4), we obtain $\omega^*_{ijkl} = \omega^*_{lkji}$, $\omega^*_{ijkl} \neq \omega^*_{ijkl}$ and $\omega^*_{ijkl} \neq \omega^*_{klij}$.

The displacement field giving solution to any problem of elastodynamics should satisfy some jump conditions on the surfaces of discontinuity similar to Eqs. (13) and (14) (see for detail, e.g. Brockway, 1972; Eringen and Suhubi, 1974; Guz, 1986a).

One can study plane sinusoidal waves of the form $u_k = a_k \exp i(L_m X_m - Ut)$ using the coordinate system X_k where U is the speed, $\mathbf{a}$ is the constant vector of amplitude and $\mathbf{L}$ is the constant vector of the direction of propagation. Substituting this expresion into Eq. (A2), one obtains the following equation

$$\left| B_{jk} - U^2\rho^{(0)}\delta_{jk} \right| = 0, \quad B_{jk} = \omega_{ijkl}L_iL_l \tag{A5}$$

to find values of U.

The value U is the wave speed refered to natural dimensions and it can be called *natural speed* for propagation normal to a plane of natural normal $\mathbf{L}$. In the system y_i, U is the wave speed referred to true dimensions and it can be called *true speed* for propagation normal to a plane of normal $\mathbf{L}$ (see, e.g. Guz, 1986b). We will denote the true speeds as C, keeping the notation U for the natural speeds. To find values of C, one should solve the following equation

$$\left| S_{jk} - C^2 \rho^{(0)} \delta_{jk} \right| = 0, \quad S_{jk} = \omega_{ijkl} L_i \lambda_i L_l \lambda_l, \tag{A6}$$

which can be obtained by substituting $u_k = a_k \exp i(L_m y_m - Ct)$ into Eq. (A3) and taking Eq. (A4) into account.

It is known (see, e.g., Hayes and Rivlin, 1961) that a real material can be maintained in the state of pure homogeneous deformation if the three roots of Eq. (A6) for C^2, or the roots of Eq. (A5) for U^2, are all positive. These are the so-called strong Hadamard conditions. These conditions can be written in a variety of forms. For example, the tensors ω_{ijkl} are strongly elliptic, i.e.,

$$\omega_{ijkl} \xi_{ji} \xi_{kl} > 0, \quad \omega^*_{ijkl} \xi_{ji} \xi_{kl} > 0 \tag{A7}$$

for arbitrary nonzero ξ_{mn}. Another form of these conditions is that the quadratic forms, $S_{jk} a_j a_k$ and $B_{jk} a_j a_k$, are positive for arbitrary nonzero a_l.

We will assume that the material under consideration satisfies the strong Hadamard conditions.

Evidently, both systems of coordinates can be used to describe the plane wave problem. The formulation of the problem for the X_k system with the use of the tensor t_{ij} is mathematically equivalent to the formulation of the problem for the y_k system with the use of the tensor Q_{ij}. The problems are considered in the paper using only the system y_k.

The formulation of the boundary-initial value problem can be closed if we use Eqs. (7)–(14), substituting y_k instead of x_k, ω^*_{ijkl} instead of C_{ijkl} and Q_{ij} instead of σ_{ij}.

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